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Splines with Maximal Zero Sets

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Consider a spline $s(x)$ of degree n with L knots of specified multiplicities $R_1, ..., R_L$, which satisfies r sign consistent mixed boundary conditions in addition to $s^{(n)}(a) = 1$. Such a spline has at most $n + 1 - r + \sum_{j=1}^{L} R_j$ zeros in (a, b) which fulfill an interlacing condition with the knots if $s(x) \not\equiv 0$ everywhere. Conversely, given a set of $n - r + \sum_{j=1}^{L} R_j$ zeros then for any choice $\eta_1 < \cdots < \eta_L$ of the knot locations which fulfills the interlacing condition with the zeros, the unique spline $s(x)$ possessing these knots and zeros and satisfying the boundary conditions is such that $s^{(n)}(x)$ vanishes nowhere and changes sign at η_j if and only if R_j is odd. Moreover there exists a choice of the knot locations, not necessarily unique, which makes $|s^{(n)}(x)| \equiv 1$. In particular, this establishes the existence of monosplines and perfect splines with knots of given multiplicities, satisfying the mixed boundary conditions and possessing a prescribed maximal zero set. An application is given to double-precision quadrature formulas with mixed boundary terms and a certain polynomial extremal problem connected with it.

1. INTRODUCTION

A polynomial spline $s(x)$ of degree n with L knots $a < \eta_1 < \cdots < \eta_L < b$ of multiplicities $R_1, ..., R_L$, $\sum_{j=1}^{L} R_j = k$, may be written as

$$s(x) = \sum_{i=0}^{n} c_i x^i + \sum_{j=1}^{L} \sum_{i=1}^{R_j} d_{ji}(x - \eta_j)_-^{n-R_j+i}, \tag{1.1}$$

where $(x)_-^n = x^n$ if $x < 0$ and zero otherwise. It will be assumed that $s(x)$ satisfies the boundary conditions

$$\sum_{j=0}^{n-1} [a_{ij}s^{(j)}(a) + b_{ij}s^{(n-1-j)}(b)] = 0, \qquad i = 1,..., r \tag{1.2}$$

on which a certain sign consistency condition is imposed (Section 2). Then the number of zeros of $s(x)$ in (a, b), $Z(s; (a, b))$ is bounded by

$$Z(s; (a, b)) \leqslant n + 1 + k - r - \sum_{j=1}^{L} \sigma_j, \tag{1.3}$$

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where $\sigma_j = 0$ if $(-1)^{R_j} s^{(n)}(\eta_j^-) s^{(n)}(\eta_j^+) > 0$ and $\sigma_j = 1$ otherwise; this bound is decreased by 1 if the boundary conditions are separate. Furthermore, if $s(x) \not\equiv 0$ in (a, b), the zeros and knots must fulfill certain interlacing conditions.

Conversely, given $n + k - r$ specified points x_i and any placement of the knots such that the interlacing condition holds, consider the unique spline $s(x)$ of degree n possessing these knots, satisfying (1.2) and

$$s(x_i) = 0, \qquad i = 1,\dots, n + k - r, \tag{1.4}$$

$$s^{(n)}(a) = 1. \tag{1.5}$$

The piecewise constant nth derivative of this spline exhibits a definite sign pattern

$$s^{(n)}(x) \prod_{j=1}^{m} (-1)^{R_j} > 0, \qquad \eta_m < x < \eta_{m+1}, \quad 1 \leqslant m \leqslant L. \tag{1.6}$$

Moreover, and this is hardest to show, there exist locations of the knots such that $| s^{(n)}(x)| \equiv 1$, and these locations are unique except when the boundary conditions are truly mixed. Even then the corresponding splines always coincide on certain intervals containing the end points. For example, the conditions $M(-1) = M(1)$, $M(x_i) = 0$ for $x_i = 0, \pm\frac{1}{3}, \pm\frac{2}{3}$ are satisfied by any monospline of degree 2 with two knots $\xi_1, \xi_2 (R_1 = R_2 = 2)$ such that $-\frac{1}{3} \leqslant \xi_1 \leqslant -\frac{1}{6}$ and $\xi_2 = \xi_1/(9\xi_1 + 1)$, and all these monosplines coincide on $(-1, -\frac{1}{3})$ and $(\frac{1}{3}, 1)$.

These are the main new results of this paper. They have obvious implications for monosplines and perfect splines. Indeed, upon taking all R_j even we obtain the existence of a monospline with multiple knots satisfying mixed boundary conditions and possessing a specified maximal zero set. In view of (1.3) this also proves existence in the case of multiplicities of arbitrary parities. In this direction our results extend those of Schoenberg [15], Karlin and Schumaker [10], and Karlin and Micchelli [8], who considered $R_j = 2$ and separated boundary conditions, and those of Karlin [5] and Micchelli [11] for multiple knots without boundary conditions. On the other hand, taking all R_j odd yields a perfect spline with multiple knots satisfying (1.2) and (1.4). Previous results in this area have concentrated on perfect splines with simple knots and the interpolation of arbitrary data [1, 6, 7].

The above intimate connection between monosplines and perfect splines also finds its expression in the dual problems in which they figure as illustrated by the following example (cf. [2, 9, 12, 16]). The unique perfect spline of degree $n = (2m + 1)L$ with all knots of multiplicity $2m + 1$ and satisfying the boundary conditions $s^{(j)}(a) = s^{(j)}(b) = 0$, $j = 0,\dots, n - 1$, $s^{(n)}(a) = 1$, has its knots located at the zeros of that polynomial $q(x)$ of degree L with highest coefficient 1 which minimizes $\int_a^b | q(x)|^{2m+1}\, dx$. On the other hand, on taking

$n = 2mL$, the knots of multiplicity $2m$ and the above boundary conditions one finds that the unique monospline satisfying these conditions has its knots located at the zeros of $q(x)$ when it minimizes $\int_a^b |q(x)|^{2m}\,dx$.

In the latter case the knots are also the nodes of the "double-precision" interpolatory quadrature formula which uses at each point $2m-1$ function and derivative values (cf. [17]). More generally, using the ideas of Schoenberg [15], our results show for instance that for given $0 < x_0 < \cdots < x_k < 1$ there exist L nodes $0 = \eta_0 < \eta_1 < \cdots < \eta_{L+1} = 1$ and corresponding weights A_{ji}, B_j such that the quadrature formula

$$\int_0^1 f(x)\,dx \sim \sum_{j=0}^{L+1} \sum_{i=0}^{R_j-2} A_{ji} f^{(i)}(\eta_j) + \sum_{j=1}^{m} B_j[f^{(j)}(0) - f^{(j)}(1)]$$

is exact for all splines of degree m with simple knots at x_i, $i = 0,\ldots, 2k$; moreover $A_{ji} > 0$, $i = 0, 2,\ldots, R_j - 2$, $j = 0,\ldots, L+1$ ($R_0 = R_{L+1} = 2$). As mentioned before the nodes are not necessarily unique in this case.

Finally it is worthwhile pointing out that our methods apply with minor modifications to Tchebycheffian splines although for simplicity we confine attention to polynomial ones.

2. Some Properties of Splines with Maximal Zero Sets

In order to be assured of the possibility of finding a spline of degree n with a total of $k = \sum_{j=1}^{L} R_j$ knots satisfying (1.2), (1.4), and (1.5), an interpolating spline for short, it is necessary to impose a sign consistency condition on the boundary form (cf. [14]). Define the matrices

$$\tilde{A}_{r,m+1} = \|(-1)^{k+n-1-j}\, a_{ij}\|_{i=1,j=0}^{r\quad m}, \qquad\qquad (2.1)$$
$$B_{r,m+1} = \| b_{ij} \|_{i=1,j=n-1-m}^{r\quad n-1}, \qquad 0 \leqslant m \leqslant n-1,$$

$$\rho(m) = \operatorname{rank} \| \tilde{A}_{r,m}, B_{r,m} \|, \qquad p = \operatorname{rank} \tilde{A}_{r,n}, \qquad q = \operatorname{rank} B_{r,n}. \qquad (2.2)$$

We assume henceforth that the following requirement holds.

Postulate I. (i) *The matrix* $\| \tilde{A}_{r,n}, B_{r,n} \|$ *is sign consistent of order* r (SC_r) *and has rank* r.

(ii) *If* $k < n$ *then*

$$n + k - r + \rho(m) \geqslant m, \qquad 1 \leqslant m \leqslant n-k. \qquad (2.3)$$

The purpose of the latter requirement, the Polya condition, is to rule out the possibility that the interpolation and boundary conditions, (1.2), (1.4), can be met simply by a polynomial. It always holds if $k \geqslant \min(r-p, r-q)$. The importance of (i) lies in the following implication.

LEMMA 2.1. *If the spline $s(x)$ of degree n satisfies the boundary conditions* (1.2) *which conform to Postulate* I, *then*

$$\begin{aligned} &S^+((-1)^i s^{(i)}(a))_0^n + S^+(s^{(i)}(b))_0^n \\ &\quad \geqslant r-1+S^+(s^{(n)}(a), -s^{(\alpha')}(a)) + S^+(-(-1)^{n-\beta'} s^{(\beta')}(b), s^{(n)}(b)) \\ &\qquad + S^+(s^{(\alpha')}(a), (-1)^{k+n-\beta'} s^{(\beta')}(b)), \end{aligned}$$

where α', β' are the largest integers $\leqslant n-1$ such that $s^{(\alpha')}(a)\, s^{(\beta')}(b) \neq 0$. If the boundary conditions are separated then the last term is replaced by 1.

Here $S^+(c_i)_0^n$ denotes the maximum number of sign changes in the sequence $c_0, \ldots, c_n$ when zero entries are replaced by $+1$, or -1. For a proof see [3]. This lemma will be used in conjunction with the Budan–Fourier theorem, which gives a bound on the number of zeros (counting multiplicities) of a spline $s(x)$ in the interval (a, b), $Z(s; (a, b))$.

LEMMA 2.2. *Let $s(x)$ be a spline of degree n exactly. Then*

$$Z(s; (a, b)) + S^+((-1)^i s^{(i)}(a))_0^n + S^+(s^{(i)}(b))_0^n \leqslant n + W(s; (a, b))$$

and the two sides differ at most by an even integer.

For the proof and the precise definition of $W(s; (a, b))$ we refer to [13]. For completeness we will in most cases provide the appropriate expression for $W(s; (a, b))$; roughly speaking it counts the number of sign changes of $s^{(n)}(x)$ taking the knots into account. We are now prepared to explore some properties of splines.

PROPOSITION 2.1. *For any spline $s(x)$ of degree n with L knots $a < \eta_1 < \cdots < \eta_L < b$ of multiplicities $R_1, \ldots, R_L$ which satisfies the boundary conditions* (1.2)

$$Z(s; (a, b)) \leqslant n + 1 + k - r - \sum_{j=1}^{L} \sigma_j, \tag{2.4}$$

where $\sigma_j = 0$ if $(-1)^{R_j} s^{(n)}(\eta_j^-)\, s^{(n)}(\eta_j^+) > 0$ and $\sigma_j = 1$ otherwise; this bound i decreased by 1 *if the boundary conditions are separated* ($r = p + q$).

Proof. We assume that $s^{(n)}(x) \neq 0$ everywhere, referring for the general cas to [13]. Then, by definition, $W(s; (a, b)) = \sum_{j=1}^{L} W(s; \eta_j)$ and

$$W(s; \eta_j) = S^+(\{(-1)^{n-R_j-i} s^{(i)}(\eta_j^-)\}_n^{n-R_j}, \{s^{(i)}(\eta_j^+)\}_{n+1-R_j}^n) - R_j,$$

whence $W(s; \eta_j) \leqslant R_j - \sigma_j$. The proof is completed by invoking Lemmas 2.1 and 2.2.

PROPOSITION 2.2. *Let $s(x)$ be as in Proposition 1.1 and suppose $s^{(n)}(a) > 0$. If $Z(s; (a, b)) \geqslant n + k - r$ and $s(x)$ does not vanish identically anywhere in (a, b) then $s^{(n)}(x)$ vanishes nowhere and*

$$\begin{aligned} &\epsilon_{j-1} s^{(i)}(\eta_j^-) \geqslant 0, \qquad \epsilon_{j-1}(-1)^{n-R_j-i} s^{(i)}(\eta_j^+) \geqslant 0, \qquad i = n - R_j, \ldots, n, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, \ldots, L \\ &s^{(n-1)}(a) \geqslant 0, \qquad\qquad (-1)^k s^{(n-1)}(b) \geqslant 0, \end{aligned} \tag{2.5}$$

where $\epsilon_j = \prod_{i=1}^{j} (-1)^{R_i}$, and the inequalities are strict unless otherwise implied by the zeros of $s(x)$. In particular in (1.1) $\epsilon_{j-1} d_{ji} > 0$, $i = 1, 3, 5, \ldots, j = 1, \ldots, L$.

Remark 2.1. Equality in (2.5) can occur for at most one j and in that case $Z(s; (a, b)) = n + 1 + k - r$. Thus for separated boundary conditions the inequalities are always strict.

Proof. From the previous proposition it follows that $\sigma_j = 0$ for all j except possibly one, say $j = l$. Were $\sigma_l = 1$, then either $(-1)^{k-1} s^{(n)}(a)\, s^{(n)}(b) > 0$ and $W(s; (a, b)) = k - 1$, or $W(s; (a, b)) \leqslant k - 2$, which on substitution in Lemmas 2.1 and 2.2 implies a contradiction, $Z(s; (a, b)) \leqslant n + k + 1 - r$. Thus each expression $W(s; \eta_j)$ has to take on its maximum value, R_j, implying (2.5).

The proof that the inequalities are usually strict is a little more delicate. It is based upon the following generalization of the Budan–Fourier theorem [13],

$$Y(s; (a, b)) + S^+((-1)^i s^{(i)}(a))_0^n + S^+(s^{(i)}(b))_0^n = n + W(s; (a, b)), \tag{2.6}$$

where $Y(s; \xi) = S^+(s^{(i)}(\xi^-))_0^n + S^+((-1)^i s^{(i)} (\xi^+))_0^n - n + W(s; \xi)$. Hence, if e.g., $s^{(l)}(\eta_j^-) = 0$ but $s^{(l-1)}(\eta_j^-)\, s^{(l+1)} (\eta_j^+) \neq 0$ for some $n + 1 - R_j \leqslant l \leqslant n - 1$ it follows from (2.5) that $Y(s; \eta_j) \geqslant Z(s; \eta_j) + 2$, while in general $Y(s; \xi) \geqslant Z(s; \xi)$, yielding a contradiction.

A further property of the spline of Proposition 2.2 is that the zeros and knots must interlace properly as required by Lemma 2.2. For example, an interval containing l knots can contain at most $n + l$ zeros. In order to formulate this condition more precisely denote by $K(\eta_i, \eta_j)$ the number of knots, counting their multiplicities, interior to (η_i, η_j) and by $N[\eta_i^+, \eta_j^-]$ the number of zeros x_i, repeated according to their multiplicities, in $[\eta_i^+, \eta_j^-]$.

INTERLACING CONDITION. *Given integers p, q, r, n such that $p \leqslant n$, $q \leqslant n$, $\max(p, q) \leqslant r \leqslant p + q$, the knots η_j, $j = 1, \ldots, L$ of respective multiplicities R_j and the points $\{x_i\}_1^{n+k-r}$ will be said to fulfill the interlacing condition if the following hold:*

1. $N(a, \eta_i^-] + r - q \leqslant n + K(a, \eta_i)$ *for all* η_i;
2. $N[\eta_i^+, b) + r - p \leqslant n + K(\eta_i, b)$ *for all* η_i;
3. $N[\eta_i^+, \eta_j^-] \leqslant n + K(\eta_i, \eta_j)$ *for all* $\eta_i < \eta_j$;
4. $N(a, \eta_i^-] + N[\eta_j^+, b) + r - 1 \leqslant 2n + K(a, \eta_i) + K(\eta_j, b)$ *for all* $\eta_i < \eta_j$.

The last relation, which is superfluous when the boundary conditions are separated, has an additional term of -1 compared to the usual case [14]. The reason for this is that the sign consistency requirement on the boundary form may only imply $S^+((-1)^i s^{(i)}(a))_0^n + S^+(s^{(i)}(b))_0^n \geqslant r - 1$, Lemma 2.1. This fact also causes the explicit formulation of the interlacing condition to be somewhat unusual, namely,

(i) either for some integer λ, $r - q \leqslant \lambda \leqslant p$,

$$x_{i-\lambda} < \xi_i < x_{i+n-\lambda}, \qquad i = 1,..., k,$$

(ii) or for some integer λ, $r - q + 1 \leqslant \lambda \leqslant p$, there exist integers i_1, i_2, $0 \leqslant i_1 < i_2 \leqslant k + 1$, such that

$$\begin{aligned} x_{i+1-\lambda} &< \xi_i < x_{i+n+1-\lambda}, & 1 &\leqslant i \leqslant i_1, \\ x_{i+1-\lambda} &< \xi_i < x_{i+n-\lambda}, & i_1 &< i < i_2, \\ x_{i-\lambda} &< \xi_i < x_{i+n-\lambda}, & i_2 &\leqslant i \leqslant k. \end{aligned}$$

Here $\{\xi_i\}_1^k$ are the same as $\{\eta_i\}_1^L$ when the latter are repeated according to their multiplicities.

As a converse to this property we have the following.

PROPOSITION 2.3. *Let* $s(x)$ *be an interpolating spline of degree* n *(i.e., satisfying* (1.2) *and* (1.4)) *whose knots* $\eta_1 < \cdots < \eta_L$ *satisfy the interlacing condition with the points* $\{x_i\}_1^{n+k-r}$. *Then* $s(x) \not\equiv 0$ *everywhere in* (a, b) *and hence the conclusions of Proposition* 2.2 *hold.*

Proof. Suppose to the contrary that, for example, $s(x) \equiv 0$ in (η_l, η_{l+1}) with $R_l \leqslant R_{l+1}$, but $s(x) \not\equiv 0$ elsewhere. Contract (η_l, η_{l+1}) to a point, by considering $\bar{s}(x)$ instead of $s(x)$ with $\bar{s}(x) = s(x)$ for $x < \eta_l$, $\bar{s}(x) = s(x + \Delta)$ for $x > \eta_l$, $\Delta = \eta_{l+1} - \eta_l$. According to the interlacing conditions one loses in the process at most n points x_i while gaining the vanishing of $\bar{s}^{(i)}(\eta_l^-)$, $i = 0,..., n - R_l$, and the loss of a knot of multiplicity R_l. Thus $Z(\bar{s}; (a, b - \Delta)) \geqslant n + 1 + k - R_l - r$, $W(\bar{s}; (a, b - \Delta)) \leqslant k - R_l$ and from Lemmas 2.1 and 2.2 it follows that the equality sign must hold in both. But then $W(\bar{s}; \eta_l) = R_{l+1}$, whence $Z(\bar{s}; \eta_l) \geqslant n + 2 - R_l$ and $Z(\bar{s}; (a, b - \Delta)) \geqslant n + 2 + k - R_l - r$, a contradiction.

In the next section it will be shown that it is possible to choose the knot locations such that the interpolating spline has $|s^{(n)}(x)| = 1$. It is clear from the interlacing condition that there is hope of obtaining this only if the knots are placed in certain intervals. In particular when the last $L - m$ knots are at permitted locations the range of η_m is restricted to (a_m, b_m), where a_m is the largest number ζ such that

$$\begin{aligned} &N[\zeta^+, b) + r - p \geqslant n + 1 + \sum_{j=m+1}^{L} R_j \\ &\text{or} \\ &\quad N[\zeta^+, \eta_l^-] \geqslant n + 1 + \sum_{j=m+1}^{l-1} R_j \qquad \text{for some } m + 1 \leqslant l \leqslant L; \end{aligned} \tag{2.6}$$

and b_m is the smallest number ζ for which at least one of the following hold.

$$N(a, \zeta^-] + r - q \geqslant n + 1 + \sum_{j=1}^{m-1} R_j,$$

$$N(a, \zeta^-] + N[\eta_l^+, b) + r - 1 \geqslant 2n + 1 + k - \sum_{j=m}^{l} R_j \qquad \text{for some } m + 1 \leqslant l \leqslant L,$$

$$\zeta = \eta_{m+1}.$$

Explicitly, let $\bar{\lambda}(\underline{\lambda})$ be the largest (smallest) integer λ, $\lambda \leqslant p$ $(\lambda \geqslant r - q + 1)$ such that $x_{i-\lambda} < \xi_i < x_{i+n+1-\lambda}$, $i = R + 1,..., k$, with $R = \sum_{j=1}^{m} R_j$. Setting $\mu = R - \bar{\lambda}$, $\nu = n + 2 + R - R_m - \underline{\lambda}$, we have

$$\begin{aligned} a_m &= x_\mu \qquad \text{if} \quad \bar{\lambda} = p \quad \text{and} \quad \xi_i < x_{i+n-p} \quad \text{for } i > R, \\ &= x_{\mu+1} \qquad \text{else}, \\ b_m &= \min(x_\nu, \eta_{m+1}). \end{aligned}$$

3. Existence and Uniqueness Properties of Splines with a Given Maximal Zero Set and nth Derivative of Constant Absolute Value

Our principal aim in this section is to establish that given $n + k - r$ zeros x_i, it is possible to choose the knots such that the interpolating spline $s(x)$ has $|s^{(n)}(x)| = 1$. By Proposition 2.2, the multiplicity of a knot determines whether $s^{(n)}(x)$ changes sign at a knot or not. For example, when all knots are even one obtains a monospline. Since moreover the maximum possible number of zeros of a monospline with a knot η_j of odd multiplicity R_j is the same as that of a monospline with η_j of even multiplicity $R_j - 1$, this result proves existence for multiplicities of arbitrary parities as well. In fact, an even stronger result holds: whenever R_j is stipulated to be odd we may actually fix the location of η_j (provided the interlacing condition holds) and still obtain a monospline. The proof of this result is virtually the same as the one we will present for the case of all multiplicities even, when use is made of the fact that we are free to add another interpolation condition, e.g., $s^{(n)}(\eta_j^+) = 1$.

Our method of proof rests upon the fact that for any choice of the knots for which the interlacing condition holds there is a unique interpolating spline $s(x)$ of degree n (i.e., such that (1.2), (1.4), and (1.5) are satisfied) possessing these knots, (cf. [14]). Proposition 2.3 shows that the interpolating spline has many properties in common with the desired spline. It remains only to show that a proper adjustment of the knots gives the desired result. This we do by induction, the induction step being the following.

THEOREM 3.1. *For fixed locations $\eta_{m+1} < \cdots < \eta_L$ of the last $L - m$ knots, restricted only by the interlacing condition, there exist locations of the first m knots $\eta_1 < \cdots < \eta_m$ such that $\{\eta_i\}_1^L$ satisfies the interlacing condition and the interpolating spline with these knots, $s(x)$, satisfies $|s^{(n)}(x)| = 1$ in (a, η_{m+1}). Moreover, the knots η_i, $i = 1,\dots, m$ and the spline $s(x)$ are uniquely determined unless there are interpolation points $\bar{x}, \bar{y} \in \{x_i\}_1^{n+k-r}$ such that $\bar{y} \leqslant \eta_{m+1}$ and*

$$N(a, \bar{x}^-] + N[\bar{y}^+, b) + r - 1 = 2n + K(a, \bar{x}) + K(\bar{y}, b).$$

In the latter case any possible nonuniqueness is confined to $(\bar{x}, \bar{y})$. In particular, uniqueness always prevails when the boundary conditions are separated.

Proof. Fixing the positions of the last $L - m$ knots, assume the theorem to be true by induction for any location η of the mth knot in the permitted range (a_m, b_m), see (2.6), (2.7). Now move the mth knot, all the while adjusting the locations of the first $m - 1$ knots, $\eta_j(\eta), j = 1,\dots, m - 1$, so as to keep on having $|s^{(n)}(x; \eta)| = 1$ in (a, η). By the induction hypothesis $s(x; \eta)$ need not be well defined in all of (a, b) but it is in (η, b). Consider

$$s_m(\eta) \equiv s^{(n)}(x; \eta) \qquad \text{for } \eta < x < \eta_{m+1}. \tag{3.1}$$

We aim to establish the induction step by showing that a location $\eta = \eta_m$ of the mth knot can be found such that $|s_m(\eta_m)| = 1$. In fact, $|s_m(\eta)| = \epsilon_m s_m(\eta)$ will be shown to be a continuous monotonic function of η, $d\,|s_m(\eta)|/d\eta \geqslant 0$, taking on all values between zero and infinity (the sign of $s_m(\eta)$ is uniquely determined; Proposition 2.2). In order to prove this we would like to consider $t(x) = (\partial/\partial\eta)\, s(x; \eta)$. If it exists everywhere it has the following properties.

(a) *$t(x)$ is a spline of degree at most n;*

(b) *$t(x)$ satisfies the interpolation and boundary conditions* (1.2) *and* (1.4); *and $t^{(n)}(a) = 0$;*

(c) *at* $\eta_j(\eta)$ $t(x)$ *has a knot of multiplicity* $R_j + 1$ *and* $t^{(n)}(\eta_j{}^+) = 0$, $j = 1,..., m - 1$; *at* η_j $t(x)$ *has a knot of multiplicity* R_j, $j = m + 1,..., L$;

(d) *at* η $t(x)$ *has a knot of multiplicity* $R_m + 1$ *and there by* (1.1) *and* (2.5),

$$\epsilon_{m-1}[t^{(n-R_m)}(\eta^+) - t^{(n-R_m)}(\eta^-)] = \epsilon_{m-1}(n + 1 - R_m)\, d_{m1} > 0. \tag{3.2}$$

More generally

$$t^{(n-R_j)}(\eta_j{}^+) - t^{(n-R_j)}(\eta_j{}^-) = (n + 1 - R_j) \frac{d\eta_j(\eta)}{d\eta} d_{j1}, \qquad j = 1,..., m - 1, \tag{3.3}$$

and $\epsilon_{j-1}\, d_{j1} > 0$.

If a unique spline with properties (a)–(d) exists it is the desired derivative. Though this is not the case in general we will see that $t(x)$ is always uniquely determined outside the first $m - 1$ knots and there then indeed $t(x) = (\partial/\partial\eta)\, s(x; \eta)$. For this purpose let us consider $u(x)$ satisfying (a)–(c) and

(d′) *at* η $u(x)$ *has a knot of multiplicity* R_m,

and show that then $u(x) \equiv 0$ in (a, η_1) and (η_{m-1}, b). This tactic may also be viewed differently on the basis of the explicit representation (1.1) of $s(x)$. Consider the system of $n + k + m$ nonlinear equations, comprised by the interpolation and boundary conditions on $s(x; \eta)$ and the conditions $s^{(n)}(x) \prod_{i=1}^{j} (-1)^{R_i} = 1$, $\eta_j < x < \eta_{j+1}$, $j = 1,..., m - 1$; involving the $n + k + m + 1$ unknowns c_i, $i = 0,..., n$, d_{ji}, $i = 1,..., R_j$, $j = 1,..., L$, η_j, $j = 1,..., m - 1$, and η. We are then about to show that either all these unknowns may be solved for in terms of η, i.e., the Jacobian does not vanish, or, if not, it can be done at least for c_i, $i = 0,..., n$, and d_{ji}, $i = 1,..., R_j$, $j = m,..., L$, i.e., the Jacobian is decomposable.

LEMMA 3.1. *If* $u(x)$ *has properties* (a)–(c) *and* (d′) *then either* $u(x) \equiv 0$ *or there exist* ζ_1, ζ_2, $\eta_1 \leqslant \zeta_1 < \zeta_2 < \eta$ *such that* $u(x) \equiv 0$ *in* (a, ζ_1) *and* (ζ_2, b) *and*

$$N(a, \zeta_1{}^-] + N[\zeta_2{}^+, b) + r - 1 = 2n + K(a, \zeta_1) + K(\zeta_2, b). \tag{3.4}$$

Remark 3.1. In the case of separated boundary conditions (3.4) can never occur. In general (3.4) indicates that the interpolation problem is decomposable in the sense that the boundary conditions and the interpolation conditions in (a, ζ_1) and (ζ_2, b) suffice to determine $s(x)$ uniquely in these intervals.

Proof. Suppose to begin with that $u(x)$ is exactly of degree n. If $u(x) \not\equiv 0$ everywhere then using (b) in Lemmas 2.1 and 2.2 yields

$$W(u; (a, b)) \geqslant k + S^+(u^{(\alpha')}(a), (-1)^{k+n-\beta'} u^{(\beta')}(b), -(-1)^k u^{(n)}(b)). \tag{3.5}$$

On the other hand (c) and (d′) may be used to provide an upper bound for $W(u; (a, b))$. For simplicity assume $u^{(n-1)}(x) \neq 0$ in (a, η), and $u^{(n)}(x) \neq 0$ in

(η, η_L); an assumption to the contrary only lowers the upper bound (cf. [12]). In this case by definition

$$\begin{aligned} W(u; (a, \eta]) &= \sum_{j=1}^{m-1} S^+(\{-(-1)^{n-R_j-i} u^{(i)}(\eta_j^-)\}_{n-1}^{n-R_j-1}, \{u^{(i)}(\eta_j^+)\}_{n-R_j}^{n-1}) \\ &\quad + S^+(\{-(-1)^{n-R_m-i} u^{(i)}(\eta^-)\}_{n-1}^{n-R_m}, \{u^{(i)}(\eta^+)\}_{n-R_m+1}^{n}) - \sum_{j=1}^{m} R_j + 1 \\ &\leqslant \sum_{j=1}^{m} R_j - S^+(u^{(n-1)}(a), \epsilon_m u^{(n)}(\eta^+)), \end{aligned}$$

$$W(u; (\eta, b)) = \sum_{j=m+1}^{L} W(u; \eta_j) \leqslant \sum_{j=m+1}^{L} R_j - S^+(u^{(n)}(\eta^+), \epsilon_m(-1)^k u^{(n)}(b)).$$

Hence

$$W(u; (a, b)) \leqslant k - 1 + S^+(u^{(n-1)}(a), -(-1)^k u^{(n)}(b)),$$

which contradicts (3.5). By the same method it is easily seen that a contradiction cannot be prevented by assuming $u(x) \equiv 0$ in a subinterval.

We conclude therefore that $u(x)$ is at most of degree $n - 1$ and its knots are of multiplicities R_j, $j = 1,..., m - 1$, and $R_j - 1$, $j = m,..., L$. If now

$$N(a, \eta_i^-] + N[\eta_j^+, b) + r \leqslant 2n + K(a, \eta_i) + K(\eta_j, b) \qquad \text{for all } \eta_i < \eta_j,$$

then the knots $\{\eta_i\}_1^L$ and points $\{x_i\}_1^{n+k-r}$ satisfy the interlacing condition necessary for unique interpolation by a spline of degree $n - 1$ [13]. Since the boundary form has the requisite sign consistency, it follows that in this case $u(x) \equiv 0$.

Assume then that ζ_2 is smallest such that (3.4) holds for some $\zeta_1 < \zeta_2$. It is readily seen that $u(x) \equiv 0$ in (a, ζ_1) and (ζ_2, b). For example, suppose to the contrary that $u(x)$ is there of degree l exactly and does not vanish anywhere. Then Lemmas 2.1 and 2.2 applied to (a, ζ_1) and (ζ_2, b) and definition (2.2) show that

$$2l + W(u; (a, \zeta_1)) + W(u; (\zeta_2, b)) \geqslant 2n - r + K(a, \zeta_1) + K(\zeta_2, b) + \rho(l)$$

with $\rho(l) \geqslant r - 2(n - 1 - l)$, which contradicts $W(u; (a, \zeta_1)) \leqslant K(a, \zeta_1)$.

Finally we want to show that $\zeta_2 < \eta$. Indeed, if this is not the case, say $\zeta_1 = \eta_1$, $\zeta_2 = \eta_L$, then the conditions $u^{(i)}(\eta_1) = 0$, $i = 0,..., n - k_1 - 1$ and $u^{(i)}(\eta_L) = 0$, $i = 0,..., n - R_L$, in addition to the vanishing of $u(x)$ at the interpolation points situated in (ζ_1, ζ_2) determine $u(x)$ uniquely, i.e., $u(x) \equiv 0$.

LEMMA 3.2. *Let $s(x; \eta)$ be an interpolating spline with knots $\eta_1 < \cdots < \eta_{m-1} < \eta < \eta_{m+1} < \cdots < \eta_L$ such that $|s^{(n)}(x; \eta)| = 1$ in (a, η). Then*

$d \mid s_m(\eta)| / d\eta \geqslant 0$; also $d\eta_j(\eta)/d\eta \geqslant 0$, $j = 1, \dots, m-1$, whenever these derivatives are defined. Moreover, $d \mid s_m(\eta)| / d\eta = 0$ if and only if $(\partial/\partial\eta)\, s(x;\eta) = 0$ in (η, b) i.e., for some $\eta_\mu < \eta$

$$N(a, \eta_\mu^-] + N[\eta^+, b) + r - 1 = 2n + K(a, \eta_\mu) + K(\eta, b), \qquad (3.6)$$

in which case $s(x;\eta)$ is uniquely determined in (a, η_μ) and (η, b) solely by the conditions applicable there.

Proof. We consider only the case that (3.4) does not hold, i.e., $t(x) = (\partial/\partial\eta)\, s(x;\eta)$ which has properties (a)–(d) is defined everywhere. By the arguments of the previous lemma it is seen that if $t(x)$ is of degree $n-1$ at most then (3.6) must hold. Assume then that $t(x)$ is of degree n exactly.

If $t(x) \not\equiv 0$ everywhere we get, as in the previous lemma,

$$W(t; (a, b)) \geqslant k + S^+(t^{(\alpha)}(a), (-1)^{k+n-\beta'}\, t^{(\beta')}(b), -(-1)^k\, t^{(n)}(b))$$

while the upper bound becomes (η is of multiplicity $R_m + 1$)

$$W(t; (a, b)) \leqslant k + S^+(t^{(\alpha)}(a), (-1)^{k+n-\beta}\, t^{(\beta)}(b)),$$

where the two sides differ at most by an even integer. Hence $\beta = n$ and $W(t; (a, b)) = k + 1$, implying that the maximum number of sign changes is achieved in each component of W. In particular, in view of $\epsilon_{m-1}[t^{(n-R_m)}(\eta^+) - t^{(n-R_m)}(\eta^-)] > 0$,

$$\text{(i)} \quad \frac{d \mid s_j(\eta)|}{d\eta} = \epsilon_j t^{(n)}(x) > 0, \qquad \eta_j < x < \eta_{j+1}, \quad j = m, \dots, L,$$

$$\text{(ii)} \quad \epsilon_{j-1}[t^{(n-R_j)}(\eta_j^+) - t^{(n-R_j)}(\eta_j^-)] > 0, \qquad \text{i.e.,} \quad \frac{d\eta_j(\eta)}{d\eta} > 0, \quad j = 1, \dots, m-1.$$

It is easily seen that even if $t(x) \equiv 0$ somewhere the above conclusions have to be modified only in so far as to allow for equality. Moreover $d \mid s_j(\eta)| / d\eta > 0$ and $ds_{j+1}(\eta)/d\eta = 0$ for some $j \geqslant m$ if and only if $t(x) \equiv 0$ in (η_{j+1}, b) and (3.6) holds with η replaced by η_{j+1}.

LEMMA 3.3. *With $s(x;\eta)$ as in the previous lemma*

$$\lim_{\eta \downarrow a_m} | s_m(\eta)| = 0 \qquad \textit{and} \qquad \lim_{\eta \uparrow b_m} | s_m(\eta)| = \infty.$$

Proof. When $\eta \to a_m$, $s(x;\eta) \to 0$ uniformly for x in (η, b). Take for instance the case that $N[a_m^+, b) + r - p \geqslant n + 1 + \sum_{j=1}^{m+1} R_j$, i.e., $a_m = x_\mu$, $\mu = \sum_{j=1}^{m} R_j - p$. Then, as $\eta \to a_m$, $s(x;\eta)$ converges in (η, b), independently of the locations of the knots η_i, $i = 1,..., m$, to the unique spline of degree n with knots $\eta_{m+1},..., \eta_L$ which interpolates to zero at $x_\mu,..., x_{n+k-r}$ and fulfills the $r - p$ boundary conditions involving only the endpoint b, i.e., to zero.

When $\eta \to b_m$ $s(x; \eta)$ converges in (a, η) and (η_l, b) for the appropriate η_l (see (2.7)). This is easily seen to be true when $\eta_i(\eta)$, $i = 1,..., m - 1$, continues to fulfill the interlacing condition even in the limit. Assume then, in view of the previous lemma, that for some j, $1 \leqslant j \leqslant m - 1$, $\eta_j \to b_j$. Then there is convergence in (a, η_j) and (η_l, b). Moreover, since $|s^{(n)}(x; \eta)| = 1$ in (η_j, η_{j+1}) and since, as we will presently see, $s^{(\gamma)}(x; \eta)$ has there at least one zero, $\gamma = n + 1 - R_j,..., n - 1$, it follows that $s(x; \eta)$ converges also in (a, η_{j+1}) and hence in (a, η). To see that $s^{(\gamma)}(x; \eta)$ vanishes in (η_j, η_{j+1}) suppose that this is not implied by the zero set. Then it is seen from Proposition 2.2 that $S^+((-1)^i s^{(i)}(\eta_j^+; \eta))_\gamma^n = 0$ and certainly $S^+(s^{(i)}(\eta_{j+1}; \eta))_{n-1}^n = 0$. Using the generalized Budan–Fourier theorem for a polynomial $p(x)$ of degree n [12],

$$Y(p; (a, b)) + S^+((-1)^i p^{(i)}(a))_0^n + S^+(p^{(i)}(b))_0^n = n,$$

where

$$Y(p; \bar{x}) = S^+(p^{(i)}(\bar{x}))_0^n + S^+((-1)^i p^{(i)}(\bar{x}))_0^n - n = Z(p; \bar{x}) + 2h,$$

h integer, we get that if $s^{(\gamma)} \neq 0$ in (η_j, η_{j+1}) then $Y(s^{(\gamma)}; (\eta_j, \eta_{j+1})) \geqslant 2$ and $Y(s; (\eta_j, \eta_{j+1})) \geqslant Z(s; (\eta_j, \eta_{j+1})) + 2$. Substitution in (2.6) yields a contradiction.

If it is now assumed by contradiction that $\lim_{\eta \uparrow b_m} |s_m(\eta)|$ is finite it follows similarly that $s(x; \eta)$ converges in (a, η_{m+1}) and (η_l, b) as $\eta \to b_m$. This however cannot be the case, for if b_m is an interpolation point it would imply $s(x; b_m^-) = 0$ in (a, b_m), contradicting $|s^{(n)}(x)| = 1$; and if $b_m = \eta_{m+1}$ convergence follows in all of (a, b) with $s(x; b_m^-)$ an interpolating spline with too few knots.

LEMMA 3.4. *Let $s(x)$ with knots $\{\eta_i\}_1^L$ and $\bar{s}(x)$ with knots $\bar{\eta}_1,..., \bar{\eta}_m, \eta_{m+1},..., \eta_L$ be two interpolating splines for which $|s^{(n)}(x)| = |\bar{s}^{(n)}(x)| = 1$ in (a, η_{m+1}). Then there are interpolation points $\bar{x}$, $\bar{y}$ such that $s(x)$ and $\bar{s}(x)$ coincide on $(a, \bar{x})$ and $(\bar{y}, b)$ and for both*

$$N(a, \bar{x}^-] + N[\bar{y}^+, b) + r - 1 = 2n + K(a, \bar{x}) + K(\bar{y}, b).$$

Proof. If $\eta_m = \bar{\eta}_m$, the lemma is true by the induction hypothesis. Suppose then that $\eta_m < \bar{\eta}_m$ and therefore $d\,|s_m(\eta)|/d\eta = 0$ for $\eta_m < \eta < \bar{\eta}_m$. Select $\bar{\eta}_\mu$, μ least, according to Lemma 3.2 and choose $\bar{x}(\bar{y})$ to be the interpolation point closest to the left (right) of $\bar{\eta}_\mu(\bar{\eta}_m)$. Then the relation of the lemma holds for $s(x)$. Moreover, $d\eta_j(\eta)/d\eta = 0$ for $1 \leqslant j \leqslant \mu - 1$ and $\eta_m < \eta < \bar{\eta}_m$. Now consider what happens when the position η of the mth knot varies from $\bar{\eta}_m$ to η_m. As long as η does not pass an interpolation point $\eta_\mu(\eta)$ cannot do so as otherwise $d\eta_u(\eta)/d\eta = 0$. Thus the relation continues to hold. It also follows that η cannot possibly pass an interpolation point if $ds_m(\eta)/d\eta = 0$ is to continue to hold for it would mean that the spline $s(x; \eta)$, which is already uniquely determined in $(a, \bar{x})$ and (η, b), has to satisfy an additional interpolation condition.

References

1. C. DE BOOR, A remark concerning perfect splines, *Bull. Amer. Math. Soc.* **80** (1974), 724–727.
2. G. GLAESER, Prolongement extrémal de fonctions différentiables d'une variable, *J. Approximation Theory* **8** (1973), 249–261.
3. S. KARLIN, "Total Positivity," Vol. I, Stanford Univ. Press, Stanford, Calif., 1968.
4. S. KARLIN, Best quadrature formulas and splines, *J. Approximation Theory* **4** (1971), 59–90.
5. S. KARLIN, On a class of best nonlinear approximation problems, *Bull. Amer. Math. Soc.* **78** (1972), 43–49.
6. S. KARLIN, Some variational problems on certain Sobolev spaces and perfect splines, *Bull. Amer. Math. Soc.* **79** (1973), 124–128.
7. S. KARLIN, Interpolation properties of generalized perfect splines, *Trans. Amer. Math. Soc.*, **106** (1975), 25–66.
8. S. KARLIN AND C. MICCHELLI, The fundamental theorem of algebra for monosplines satisfying boundary conditions, *Israel J. Math.* **11** (1972), 405–451.
9. S. KARLIN AND A. PINKUS, *Gaussian quadrature formulae with multiple nodes*, in "Studies in Spline Functions and Approximation Theory," S. Karlin, C. A. Mocchelli, A. Pinkus, and I. J. Schoenberg eds., Academic Press, New York, 1977.
10. S. KARLIN AND L. SCHUMAKER, The fundamental theorem of algebra for Tchebycheffian monosplines, *J. Analyse Math.* **20** (1967), 233–270.
11. C. MICCHELLI, The fundamental theorem of algebra for monosplines with multiplicities, Proc. Conf. Oberwolfach 1971, *ISNM* **20** (1972), 419–430.
12. C. A. MICCHELLI AND T. J. RIVLIN, Quadrature formulae and Hermite–Birkhoff interpolation, *Advances in Math.* **11** (1973), 93–112.
13. A. A. MELKMAN, The Budan–Fourier theorem for splines, *Israel J. Math.* **19** (1974), 256–264.
14. A. A. MELKMAN, Interpolation by splines satisfying mixed boundary conditions, *Israel J. Math.* **19** (1974), 369–382.
15. I. J. SCHOENBERG, Spline functions, convex curves and mechanical quadrature, *Bull. Amer. Math. Soc.* **69** (1958), 352–357.
16. I. J. SCHOENBERG, The perfect B-splines and a time optimal control problem, *Israel J. Math.* **10** (1971), 275–291.
17. P. TURAN, On the theory of mechanical quadrature, *Acta Sci. Math. (Szeged) A* **12** (1950), 30–37.